TRINITY COLLEGE

Year 12 Mathematics Specialist 3/4
Test 5 2022
Weighting 7%

Calculator Assumed
Rates of Change and Differential Equations

STUDENT'S NAME ______________________________

DATE: Thursday 18 August **TIME:** 50 minutes **MARKS**: 51

INSTRUCTIONS:

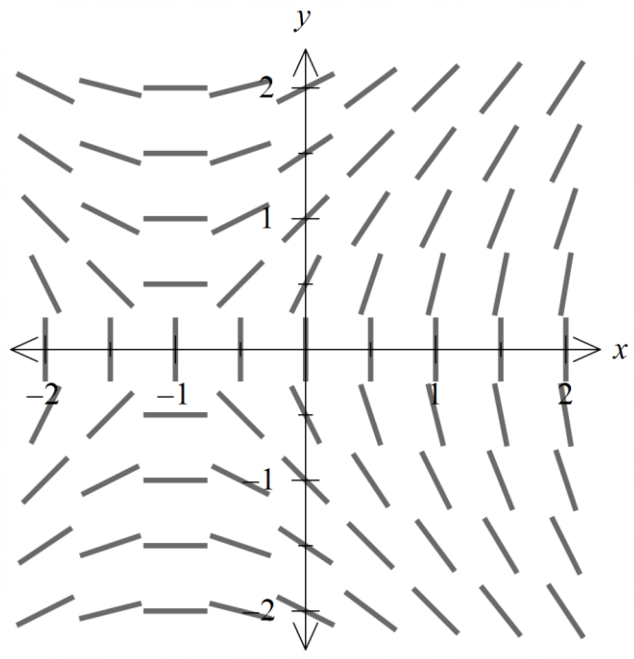
Standard Items: Pens, pencils, drawing templates, eraser
Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

This page has been left blank intentionally

1. (10 marks)

The slope field given by $\frac{dy}{dx} = \frac{x+1}{y}$ is shown below.



(a) Determine the value of the slop field at the point

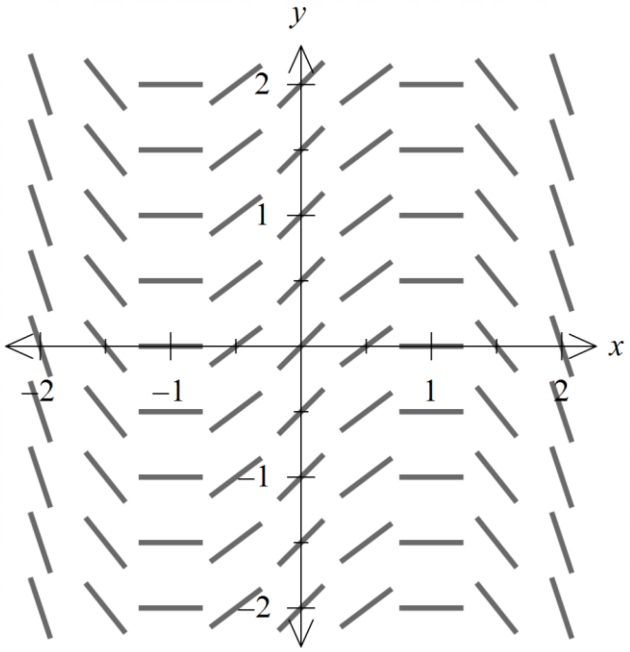
(i) $(1, 1)$ [1]

(ii) $(-1, 0)$ [1]

(b) On the diagram above, draw the solution curve that contains the point $(1, 1)$. [2]

(c) Determine the equation of the solution curve that contains the point $(1, 1)$. [3]

The slope field is adjusted. The new slope field generated is below.



(d) Determine the equation of the slope field given that at the point $(0, 0)$ the slope is 1. [3]

2. (10 marks)

A population after t years is modelled by the differential equation

$$\frac{dP}{dt} = 1.2P\left(1 - \frac{P}{4200}\right)$$

The initial value of the population was 100.

(a) Rewrite the logistic equation in the form $\frac{a}{1+be^{-ct}}$, clearly stating the values for a, b and c. [3]

(b) Determine the rate of population growth when the population is 1500. [1]

(c) Determine how long it takes for the rate of population growth to reach a maximum. [3]

(d) When the population reaches 4000, use the technique of increments to calculate the approximate change in population in the next month. [3]

3. (9 marks)

When a body, moving along the x-axis, has displacement x from the origin, its velocity v satisfies the equation

$$\frac{d}{dx}(v^2) = -18x$$

Given that $v = 4$ when $x = 1$,

(a) (i) Show that the velocity of the body is of the form $v = \sqrt{-9x^2 + c}$ [1]

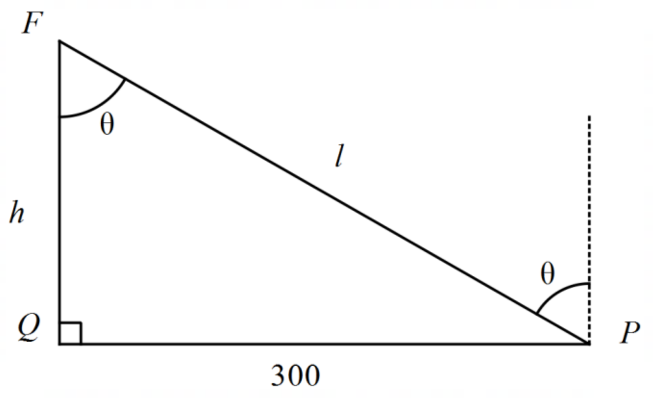
(ii) Determine the velocity of the body when $x = 0$. [2]

(b) Determine the maximum displacement of the body from the origin. [3]

(c) Determine the acceleration of the body when $x = 1$. [3]

4. (14 marks)

Light from a flare F shines on a small plate P which lies on a horizontal plane. The intensity I of the light is directly proportional to $\frac{cos\theta}{l^2}$, where l is the length of the distance form F to P, and θ is the angle of incidence as shown in the diagram below.



At time t seconds the flare is h metres above the point Q on the plane, and P is 300 metres from Q. It can be shown that $I = k\ cos\theta\ sin^2\theta$ for some constant k.
Given that $I = 72$ when $h = 400$:

(a) Evaluate k [2]

(b) Determine an expression for $\frac{dI}{d\theta}$. [2]

(c) Determine, to the nearest metre, the height h at which the intensity I is the greatest. [2]

(d) Determine the maximum value of the intensity I. [1]

The flare is falling vertically at a constant rate of 5 metres per second.

(e) Show that $\frac{d\theta}{dt} = \frac{1}{60} sin^2\theta$. [4]

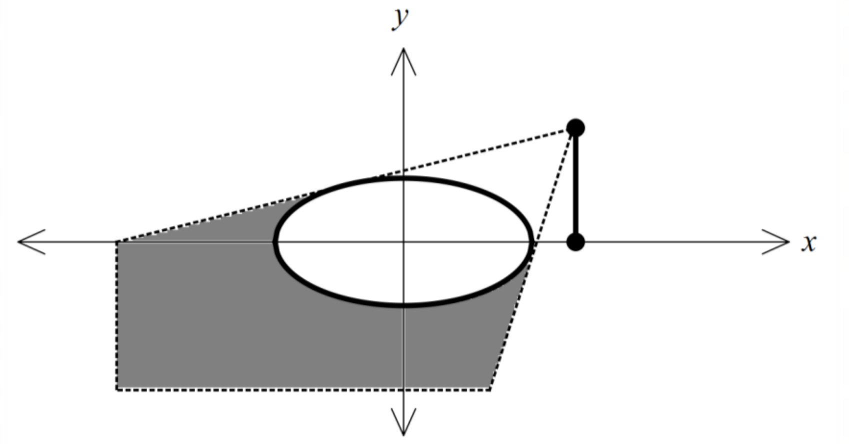
(f) Determine $\frac{dI}{dt}$ when the flare is 125 metres above the point Q. [3]

5. (8 marks)

Consider the elliptical region $x^2 + 4y^2 = 5$.

(a) Determine the slope to the ellipse in the first quadrant when $x = 2$. [3]

A lamp is located at $x = 3$. A shadow is created by the ellipse as shown in the diagram below.



(b) If the point $(-5, 0)$ is on the edge of the shadow, determine how far above the x-axis the lamp is located. [5]